NOTES ON SOME POINTS IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.

By PROF. MAXIME BÔCHER, Cambridge, Mass.

In the following pages I have tried to treat some points in the theory of linear differential equations in a simpler manner than is ordinarily done, and to insist upon some matters which are usually passed over in silence. In doing this I have not hesitated to confine myself to equations of the second order:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0\,, \tag{1}$$

as everything which is here given can be immediately extended by the same method to equations of higher order. Much of what follows was given at the Buffalo Colloquium held under the auspices of the American Mathematical Society in September, 1896 (cf. Bull. Amer. Math. Soc., Nov., 1896).

§ 1. *Equations with Analytic Coefficients. Non-Singular Points.*

We will begin by considering the case in which the coefficients p and q of the differential equation (1) are analytic functions of the complex variable x. By a non-singular point of such an equation is meant a point at which $p(x)$ and $q(x)$ are analytic,* i. e. about which they can be developed in the form:

$$p(x) = p_0 + p_1(x-a) + p_2(x-a)^2 + \ldots,$$

$$q(x) = q_0 + q_1(x-a) + q_2(x-a)^2 + \ldots.$$

We will suppose that each of these series converges when $|x-a| < R$. We wish to prove the following fundamental EXISTENCE THEOREM:

There exists a solution of the differential equation (1) *of the form:*

$$y = g_0 + g_1(x-a) + g_2(x-a)^2 + \ldots \tag{2}$$

where g_0 and g_1 are arbitrary constants and where the series converges when $|x-a| < R$.

If we let:

$$f(x, \rho) = \rho(\rho-1) + \rho(x-a)\,p(x) + (x-a)^2\,q(x)\,,$$

* It will be seen that I speak here and in the following sections of a function as analytic at a point if it can be developed in a power series about this point. It will then be analytic throughout a region if it is analytic at each point of the region. The ordinary usage is to restrict the word *analytic* to regions and to use the word *regular* for points. One word seems quite sufficient here.

the result of substituting (2) in (1) is :

$$\sum_{\nu=0}^{\nu=\infty} g_\nu f(x, \nu)\,(x-a)^{\nu-2} = 0\,. \tag{3}$$

Now we have :

$$f(x, \rho) = \sum_{\mu=0}^{\mu=\infty} f_\mu(\rho)\,(x-a)^\mu\,,$$

where $f_0(\rho) = \rho\,(\rho - 1)$, $f_1(\rho) = \rho p_0$, and when $\mu > 1$ $f_\mu(\rho) = \rho p_{\mu-1} + q_{\mu-2}$. Let us substitute this development of $f(x, \rho)$ in (3) and collect the terms involving like powers of $x - a$. If we then equate the coefficients of the different powers of $x - a$ to zero we get :

$$g_0 f_0(0) = 0\,,$$

$$g_1 f_0(1) + g_0 f_1(0) = 0\,,$$

$$g_2 f_0(2) + g_1 f_1(1) + g_0 f_2(0) = 0\,,$$

$$g_3 f_0(3) + g_2 f_1(2) + g_1 f_2(1) + g_0 f_3(0) = 0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

the first two of these equations are satisfied by any values of g_0 and g_1 since $f_0(0) = f_0(1) = f_1(0) = 0$. By means of the remaining equations we can compute in succession $g_2, g_3, \ldots$, the general formula being :

$$g_\nu = \frac{-1}{\nu\,(\nu-1)}\,[g_{\nu-1} f_1(\nu-1) + g_{\nu-2} f_2(\nu-2) + \ldots + g_0 f_\nu(0)].$$

It remains to prove that the series (2) in which the coefficients have been thus determined converges when $|x - a| < R$. We will suppose that x is any quantity satisfying this inequality and we will take x' as a second quantity satisfying the relation

$$|x - a| < |x' - a| < R\,.$$

For the sake of brevity we will let $|x' - a| = K$.

Consider the developments :

$$(x' - a)\,p\,(x') = p_0\,(x' - a) + p_1\,(x' - a)^2 + \ldots,$$

$$(x' - a)^2\,q\,(x') = q_0\,(x' - a)^2 + q_1\,(x' - a)^3 + \ldots.$$

Since these series are both absolutely convergent the absolute values of the terms will all be smaller than some positive constant M :

$$|p_{\mu-1}|\,K^\mu < M\,, \quad |q_{\mu-2}|\,K^\mu < M\,.$$

We thus get:

$$|f_\mu(\rho)| \leqq |\rho| |p_{\mu-1}| + |q_{\mu-2}| < MK^{-\mu}(|\rho| + 1),$$

$$|g_\nu| < \frac{M}{\nu(\nu-1)}[|g_{\nu-1}| K^{-1}\nu + |g_{\nu-2}| K^{-2}(\nu-1) + \ldots + |g_0| K^{-\nu}].$$

Let us call the second member of this inequality c_ν. If we can prove that the series $\Sigma c_\nu |x-a|^\nu$ is convergent our theorem is established. Now we have:

$$c_{\nu+1} = \frac{M}{(\nu+1)\nu}[|g_\nu| K^{-1}(\nu+1) + |g_{\nu-1}| K^{-2}\nu + \ldots + |g_0| K^{-\nu-1}],$$

$$\frac{c_{\nu+1}}{c_\nu} = \frac{\nu-1}{\nu+1} K^{-1} + \frac{MK^{-1}}{\nu} \frac{|g_\nu|}{c_\nu},$$

from which, remembering that $|g_\nu| < c_\nu$, it follows that:

$$\lim_{\nu=\infty} \left[\frac{c_{\nu+1}}{c_\nu} |x-a|\right] = \frac{|x-a|}{K} < 1,$$

and the series $\Sigma c_\nu |x-a|^\nu$ converges.

The proof just given is essentially an application to the case of a non-singular point of the more general proof given by Frobenius in Crelle, Vol. 76. It may be noticed, however, that as here given the proof applies to the case in which p and q are real analytic functions of the real variable x without any reference whatever to imaginary values.

§ 2. *Equations with Analytic Coefficients have none but Analytic Solutions.*

It follows at once from the algorithm of the last section that the differential equation can have no solution, other than those there discussed, which is analytic at a.* This does not, however, prove that the solutions obtained in the last section are the only analytic solutions; there might, for instance, be other analytic solutions with a singular point at a. In order to consider this question we will introduce the idea of the analytic continuation of the solutions obtained in § 1. We thus easily deduce the theorem (cf. Heffter's Einleitung in die Theorie der linearen Differentialgleichungen, § 39): *A solution of equation* (1) *that is analytic in any part of a connected region throughout which* $p(x)$ *and* $q(x)$ *are analytic, is itself analytic throughout this region.* Thus the solutions above discussed include all analytic solutions of (1).†

---

* Cf. foot-note, p. 45.

† More accurately they include all solutions analytic at any part of the connected region including a throughout which both p and q are analytic. The region throughout which p and q are both analytic may, however, consist of two or more separate pieces in which case solutions of the equation analytic in one piece would not in general exist in the others. In such a case, however, we practically have a number of distinct differential equations.

Let us, however, look at the case in which p and q are real functions of the real variable x. Then, even though p and q are analytic, there is no reason *a priori* why the differential equation should not have, besides the analytic solutions obtained in the last section, other solutions which are not analytic. It is therefore of fundamental importance to establish the following theorem :

If throughout an interval AB of the x-axis the coefficients p and q of equation (1) *are real analytic functions of the real variable x, a real function y which at every point of AB satisfies equation* (1) *will be analytic throughout AB.*

It should be noticed that when we require that the function y should satisfy the differential equation at every point of AB we thereby require that it should have a first and a second derivative at every such point, and therefore, in particular, that y and its derivative y' should be continuous throughout AB.

In order to prove the theorem just stated let a be any point of the interval AB. Call the values which y and y' have at a α and β respectively. Consider now the analytic solution whose existence was established in § 1 :

$$\bar{y} = \alpha + \beta(x - a) + g_2(x - a)^2 + \ldots$$

$y - \bar{y}$ is then a solution of the differential equation which together with its first derivative vanishes when $x = a$. Our proof will be complete if we can prove that $y - \bar{y}$ vanishes at every point of AB for we should then have $y = \bar{y} =$ an analytic solution.

It remains then to prove a theorem which can be proved with the same ease in the following more general form in which we do not require that p and q be analytic, but merely that they be continuous :

If throughout an interval AB p and q are continuous real functions of the real variable x, a real function y which satisfies equation (1) *at every point of AB and which together with its first derivative vanishes at a point a of the interval AB will vanish throughout AB.*

We will suppose that $p(x)$ and $q(x)$ are continuous at A and B as well as between these points.* It will then be possible to find a positive quantity M greater than the greatest numerical value of $p(x)$ and $q(x)$ in AB. We will suppose M to be also taken greater than $\frac{1}{4}$:

$$M > |p(x)|, \quad M > |q(x)|, \quad M > \tfrac{1}{4}.$$

We will consider first not the whole interval but only the interval from

* If this were not the case we should merely have to take two other points A' and B' between A and B and arbitrarily near to A and B respectively, and then to prove the theorem for the interval $A'B'$.

$a - 1/4M$ to $a + 1/4M$ or so much of this interval as lies between A and B. We assume that $y = 0$ and $y' = 0$ when $x = a$ and we wish to prove that $y = 0$ for all values of x between $a - 1/4M$ and $a + 1/4M$.

Consider the maxima of the functions $|y|$ and $|y'|$ in the interval just mentioned, and call the larger of these two quantities c, so that when

$$a - 1/4M \leqq x \leqq a + 1/4M \qquad |y| \leqq c \text{ and } |y'| \leqq c.$$

It will then be sufficient if we can prove that $c = 0$. It follows from the differential equation that for all points of the interval we are considering:

$$|y''| \leqq |p(x)|\,|y'| + |q(x)|\,|y| \leqq 2cM.$$

Now we see by applying the law of the mean* to the function y' that at any point x of our interval $(y' - 0)/(x - a)$ is equal to the value of y'' at some point between a and x. Accordingly:

$$|y'| \leqq 2Mc\,|x - a| \leqq c/2 \text{ (since } |x - a| \leqq 1/4M).$$

Applying the law of the mean to the function y we see that $(y - 0)/(x - a)$ is equal to the value of y' at some point between a and x, so that:

$$|y| \leqq \frac{c}{2}\,|x - a| \leqq \frac{c}{2}\cdot\frac{1}{4M} \leqq \frac{c}{2} \text{ (since } M > \tfrac{1}{4}).$$

We have thus shown that neither $|y|$ nor $|y'|$ exceeds the value $c/2$ at any point of our interval, while by hypothesis either $|y|$ or $|y'|$ has the value c at some point of this interval. It follows that $c \leqq c/2$, so that, since c cannot be negative, $c = 0$.

Having thus proved that y and y' vanish throughout the interval from $a - 1/4M$ to $a + 1/4M$ † we now proceed to extend our result to the whole interval AB. Since y is a solution of (1) which together with its first derivative vanishes when $x = a + 1/4M$, the reasoning just used shows that it must vanish throughout the interval from $a + 1/4M$ to $a + 2/4M$. Applying the same reasoning again we find that it vanishes from $a + 2/4M$ to $a + 3/4M$, etc. On the other hand, since y and y' vanish when $x = a - 1/4M$ we see that y vanishes from $a - 1/4M$ to $a - 2/4M$, from $a - 2/4M$ to $a - 3/4M$, etc. It follows that y vanishes throughout AB.

The proof just given is an application to the case of linear differential

* I. e. the theorem that if $f(x)$ is a continuous function and has a derivative: $(f(x) - f(a))/(x - a) = f'(\xi)$ where ξ is some point between a and x.

† Here and in the next few lines only so much of the intervals is meant as is included in AB.

equations of the proof given by Jordan in the revised edition of his Cours d'Analyse, Vol. III, p. 93.*

§ 3. *Equations with Real Coefficients. Existence Theorem.*

The last theorem established applies to equations in which p and q are any real continuous functions of the real variable x. In this section we will establish the following existence theorem for such equations analogous to the one established in § 1 for equations with analytic coefficients.

If in the interval AB p and q are continuous real functions of the real variable x, a real function y exists which satisfies equation (1) *at every point of AB and which has at the arbitrarily chosen point a of AB the arbitrary value α while its derivative has at this point the arbitrary value β.*

We will establish this theorem by the method of successive approximations first used for this purpose by Peano in 1887.† As a first approximation y_0 we will take the simplest function which satisfies the initial conditions $y_0(a) = \alpha$, $y_0'(a) = \beta$, i. e. the linear function :

$$y_0 = \alpha + \beta(x - a).$$

We then compute a second approximation y_1 from the relation :

$$y_1'' + p(x)\,y_0' + q(x)\,y_0 = 0\,,$$

and the initial conditions $y_1(a) = \alpha$, $y_1'(a) = \beta$. This gives when we remember that $y_0' = \beta$:

$$y_1' - y_0' = -\int_a^x [p(x)y_0' + q(x)\,y_0]\,dx\,,$$

and integrating again :

$$y_1 - y_0 = \int_a^x (y_1' - y_0')\,dx\,.$$

Proceeding in this way we compute each approximation y_n from the preceding approximation y_{n-1} by the relation :

$$y_n'' + p(x)\,y_{n-1} + q(x)\,y_{n-1} = 0$$

and the initial conditions :

$$y_n(a) = \alpha\,, \quad y_n'(a) = \beta\,.$$

* The form in which Jordan gives this proof is far from satisfactory, but his proof may readily be made rigorous. Cf. also Lindelöf : Journal de Mathématique, 1894, p. 118.

† Cf. Math. Ann. Bd. 32. For other references see the *Bulletin*, loc. cit.

This gives :

$$y_n' - y_0' = -\int_a^x [p(x)\, y_{n-1} + q(x)\, y_{n-1}]\, dx\,,$$

$$y_n - y_0 = \int_a^x (y_n' - y_0')\, dx\,.$$

We will now prove : 1) that as n increases indefinitely y_n approaches a definite limit y; 2) that this limit y satisfies the desired initial conditions; 3) that y satisfies the differential equation. Our theorem will then be established.

The problem will have a more familiar form if y_n is regarded as the sum of the first $n + 1$ terms of a series. This can be done by letting $Y_1 = y_1 - y_0$, $Y_2 = y_2 - y_1, \ldots, Y_n = y_n - y_{n-1} \ldots$. Then y_n is the sum of the first $n + 1$ terms of the series

$$y_0 + Y_1 + Y_2 + Y_3 + \ldots, \tag{4}$$

and our first problem is to prove that this series converges at all points of the interval AB. It is easy to so arrange our work as to prove at the same time the convergence of the second series :

$$y_0' + Y_1' + Y_2' + Y_3' + \ldots. \tag{5}$$

The functions $Y_1', Y_1, \ldots, Y_n', Y_n$ are computed by the formulæ :

$$Y_1' = -\int_a^x [p(x)\, y_0' + q(x)\, y_0]\, dx\,,$$

$$Y_1 = \int_a^x Y_1' dx\,,$$

$$Y_n' = -\int_a^x [p(x)\, Y_{n-1}' + q(x)\, Y_{n-1}]\, dx\,,$$

$$Y_n = \int_a^x Y_n' dx\,.$$

Let C be a positive quantity satisfying the inequalities $|\, y_0 \,| < C \; |\, y_0' \,| < C$ throughout AB.

Let $t = |\, x - x_0 \,|$ and let l be a positive constant greater than 1 and satisfying for all points of AB the inequality $t < l$.

Let M be a positive constant satisfying throughout AB the inequalities $|\, p(x) \,| < M$, $|\, q(x) \,| < M$.

We have then evidently :

$$| Y_1' | \leqq \int_0^t 2CMdt = 2CMt ,$$

$$| Y_1 | \leqq \int_0^t 2CMtdt = \frac{2CMt^2}{2} .$$

The two expressions which we have here found, one greater than $| Y_1' |$ and the other greater than $| Y_1 |$ may clearly be replaced by one and the same quantity $2CMlt$, which is greater than either of them. We thus get

$$| Y_1' | < 2CMlt , \quad | Y_1 | < 2CMlt .$$

Proceeding in the same way :

$$| Y_2' | \leqq \int_0^t 2^2 CM^2 ltdt = \frac{2^2 CM^2 lt^2}{2\,!} ,$$

$$| Y_2 | \leqq \int_0^t \frac{2^2 CM^2 lt^2}{2!} dt = \frac{2^2 CM^2 lt^3}{3!} .$$

The two expressions on the right hand side may here again be replaced by a single one :

$$| Y_2' | < \frac{2^2 CM^2 l^2 t^2}{2\,!} , \quad | Y_2 | < \frac{2^2 CM^2 l^2 t^2}{2\,!} .$$

At the next step we get :

$$| Y_3' | < \frac{2^3 CM^3 l^3 t^3}{3\,!} , \quad | Y_3 | < \frac{2^3 CM^3 l^3 t^3}{3!} ,$$

etc. It is clear then that the absolute values of the terms of the series (4) and (5) are less than the corresponding terms of the series :

$$C + C \,.\, 2Mlt + C \frac{(2Mlt)^2}{2!} + C \frac{(2Mlt)^3}{3!} + \dots ,$$

and this being a convergent series of positive terms the series (4) and (5) are absolutely convergent throughout the interval AB. Moreover the terms of the series last written being smaller than the corresponding terms of the series :

$$C + C \,.\, 2Ml^2 + C \frac{(2Ml^2)^2}{2!} + C \frac{(2Ml^2)^3}{3!} + \dots ,$$

and this being a convergent series of constant terms (i. e. terms independent

of t) it follows at once that the series (4) and (5) are uniformly convergent throughout AB. Series (5) represents therefore the derivative of (4).

It follows now at once that y satisfies the initial conditions we wish it to satisfy; for we have:

$$y = \lim_{n=\infty} y_n \text{ and } y' = \lim_{n=\infty} y'_n$$

and $y_n(a) = \alpha$, $y'_n(a) = \beta$, so that $y(a) = \alpha$, $y'(a) = \beta$.

It remains then merely to prove that y satisfies the differential equation at every point of AB. Now we have:

$$y'_n - y'_0 = -\int_a^x [p(x)y'_{n-1} + q(x)y_{n-1}]\,dx\,.$$

Let us here take the limit of each side as n becomes infinite, remembering that since y_{n-1} and y'_{n-1} approach their limits uniformly we have a right to take the limit under the sign of integration:

$$y' - y'_0 = -\int_a^x [p(x)y' + q(x)y]\,dx\,.$$

When we differentiate this equation we get:

$$y'' = -p(x)y' - q(x)y\,,$$

i. e. y satisfies the differential equation at every point of AB.

By the method used in § 2 it follows at once that no other real functions y than those just obtained exist which satisfy equation (1) at every point of AB.

Finally we may note that the exponential series obtained in the above proof of convergence may be conveniently used, if we wish to employ the method here explained for numerical computation to determine how large n must be taken in order that y_n should be a sufficiently close approximation for the purpose in hand.

HARVARD UNIVERSITY.